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THE CANONICAL SPECTRAL SEQUENCES FOR POISSON MANIFOLDS*

BY

MARISA FERNÁNDEZ AND RAÚL IBÁÑEZ

Departamento de Matemáticas, Facultad de Ciencias
Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain
e-mail: mtpferol@lg.ehu.es mtpibtor@lg.ehu.es

AND

MANUEL DE LEÓN

Instituto de Matemáticas y Física Fundamental
Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain
e-mail: mdeleon@fresno.csic.es

ABSTRACT

For a compact symplectic manifold M of dimension $2n$, Brylinski proved that the canonical homology group $H_k^{can}(M)$ is isomorphic to the de Rham cohomology group $H^{2n-k}(M)$, and the first spectral sequence $\{E^r(M)\}$ degenerates at $E^1(M)$. In this paper, we show that these isomorphisms do not exist for an arbitrary Poisson manifold. More precisely, we exhibit an example of a five-dimensional compact Poisson manifold M^5 for which $H_1^{can}(M^5)$ is not isomorphic to $H^4(M^5)$, and $E^1(M^5)$ is not isomorphic to $E^2(M^5)$.

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1. Introduction

For any Poisson manifold M there is ([3], [11], [21]) the **canonical complex**

$$\cdots \longrightarrow \Lambda^{k+1}(M) \xrightarrow{\delta} \Lambda^{k}(M) \xrightarrow{\delta} \Lambda^{k-1}(M) \longrightarrow \cdots .$$

Here $\Lambda^k(M)$ is the space of the differential k-forms on M, and δ is the differential introduced by Koszul [11], which is given by $\delta = [i(G), d]$, where G is the Poisson tensor defined by Lichnerowicz [15], $i(G)$ denotes the interior product by G and d is the exterior differential.

The homology of the canonical complex is called the **canonical homology** of M; it was extensively studied by Brylinski [3]. There Brylinski noticed that there exists the double complex $\mathcal{E}^{per}_{*,*}(M)$ on M defined by $\mathcal{E}^{per}_{p,q}(M) = \Lambda^{q-p}(M)$ for all $p, q \in \mathbb{Z}$, which has d for horizontal differential and δ for vertical differential, both of degree -1.

In the present paper we study the two spectral sequences $\{E^r(M)\}$ and $\{'E^r(M)\}$ associated with the double complex.

In Section 2, we prove that the second spectral sequence degenerates at $'E^1(M)$ for any Poisson manifold M, that is, $'E^1(M) \cong {}'E^\infty(M)$.

In the case of a compact symplectic manifold M, we also show, in Section 2, that there is an isomorphism of homology groups $E^r_{p,q}(M) \cong {}'E^r_{q,2n+p}(M)$. In particular, we deduce the following Brylinski's result ([3], Theorem 2.3.1): For a compact symplectic manifold M, the first spectral sequence $\{E^r(M)\}$ degenerates at $E^1(M)$. Furthermore, in [3] it is proved that there is an isomorphism $H^{can}_k(M) \cong H^{2n-k}(M)$, for any compact symplectic manifold of dimension $2n$.

In Section 5, we obtain that the previous Brylinski's results are not true, in general, for any compact Poisson manifold. More precisely, we give an example of a five-dimensional compact Poisson manifold M^5 for which $E^1(M^5) \not\cong E^2(M^5)$, and $H^{can}_1(M^5) \not\cong H^4(M^5)$. This manifold is a cosymplectic manifold.

Cosymplectic manifolds are another important class of Poisson manifolds. Roughly speaking, cosymplectic manifolds may be considered as the odd-dimensional counterpart of the symplectic manifolds.

For any Poisson manifold M, the term $E^1(M)$ of the first spectral sequence $\{E^r(M)\}$ is the canonical homology of M. The canonical homology groups have dim $< \infty$ for any compact symplectic manifold. In Section 3, we obtain that the same conclusion does not hold for any compact cosymplectic manifold.

The main problem to exhibit examples of compact Poisson manifolds M for which $E^1(M) \not\cong E^2(M)$ is the difficulty to compute the canonical homology of M.

For any compact nilmanifold $\Gamma\backslash K$ it is well-known a theorem due to Nomizu [19], which asserts that the Chevalley–Eilenberg cohomology $H^*(\mathfrak{K}^*)$ of the Lie algebra $\mathfrak{K}$ of K is isomorphic to the de Rham cohomology $H^*(\Gamma\backslash K)$.

In Section 4 we obtain an "aproximation to Nomizu theorem" for the canonical homology $H_*^{can}(\Gamma\backslash K)$ of any compact cosymplectic nilmanifold. In fact, we prove that there is an injective homomorphism of $H_*^{can}(\mathfrak{K}^*)$ into $H_*^{can}(\Gamma\backslash K)$. We use this result, in Section 5, to show that $E^1(M^5) \not\cong E^2(M^5)$.

2. Degeneration of the second spectral sequence

Let us recall the construction of the (canonical) double complex due to Brylinski [3].

Let M be a C^∞ manifold. Denote by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M, and by $\mathfrak{F}(M)$ the algebra of C^∞ functions on M. A **Poisson bracket** $\{,\}$ on M is a bilinear mapping

$$\{,\}\colon \mathfrak{F}(M) \times \mathfrak{F}(M) \longrightarrow \mathfrak{F}(M)$$

satisfying the following properties:

(i) $\{f, gh\} = \{f,g\}h + g\{f,h\}$,

(ii) $\{f,g\} = -\{g,f\}$,

(iii) $\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$ (**Jacobi's identity**),

for $f, g, h \in \mathfrak{F}(M)$.

For fixed $f \in \mathfrak{F}(M)$, property (i) implies that the mapping $g \mapsto \{f,g\}$ defines a vector field X_f, which is called the **Hamiltonian vector field corresponding to** f. Thus, $X_f(g) = \{f,g\}$, for $g \in \mathfrak{F}(M)$. The manifold M endowed with a Poisson bracket is called a **Poisson manifold**. Lichnerowicz in [15] remarked that the existence of a Poisson bracket is equivalent to the existence of a skew-symmetric tensor field of type $(2,0)$ on M such that

$$G(df, dg) = \{f,g\}, \tag{1}$$

for $f, g \in \mathfrak{F}(M)$, and satisfying $[G,G] = 0$, where $[,]$ is the Schouten–Nijenhuis bracket. G is called a **Poisson tensor**. (The rank of G is called the rank of the Poisson structure; in general, the rank of G is not constant.)

The local structure of a Poisson manifold is given in the following

THEOREM 2.1 ([14, 22]): *Let M be a Poisson manifold of dimension m, with Poisson bracket $\{,\}$. Let x be a point of M where the rank of the Poisson structure is $2r$. Then, there exist local coordinates*

$$\{q^1, \ldots, q^r, p_1, \ldots, p_r, z^1, \ldots, z^{m-2r}\}$$

around x such that

$$\{q^i, q^j\} = 0, \quad \{q^i, p_j\} = \delta_i^j, \quad \{q^i, z^a\} = 0, \quad \{p_i, p_j\} = 0, \quad \{p_i, z_a\} = 0,$$

for all $1 \leq i, j \leq r, 1 \leq a \leq m - 2r$. Furthermore, the Poisson bracket $\{z^a, z^b\}$ is a function only of the local coordinates $z^1, \ldots, z^{m-2r}$ and vanishes at x. If the rank of the Poisson structure is constant and equal to $2r$, the z-coordinates satisfy $\{z^a, z^b\} = 0$, for all $1 \leq a, b \leq m - 2r$. (Coordinates $\{q^1, \ldots, q^r, p_1, \ldots, p_r, z^1, \ldots, z^{m-2r}\}$ are called Darboux coordinates.)

Next, we shall denote by $\Lambda^k(M)$ the space of the differential k-forms on M. Koszul [11] introduced the differential operator δ: $\Lambda^k(M) \longrightarrow \Lambda^{k-1}(M)$ given by the commutator of $i(G)$ and the exterior differential d, that is,

$$\delta = [i(G), d] = i(G) \circ d - d \circ i(G). \tag{2}$$

On the other hand, Brylinski [3] has proved that δ can be alternatively defined by the formula

$$\begin{aligned} \delta(f_0\, df_1 \wedge \cdots \wedge df_k) = & \sum_{1 \leq i \leq k} (-1)^{i+1} \{f_0, f_i\}\, df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_k \\ & + \sum_{1 \leq i < j \leq k} (-1)^{i+j} f_0\, d\{f_i, f_j\} \wedge df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_k. \end{aligned} \tag{3}$$

From (2) or (3) one can check that $\delta^2 = 0$ (see [3], [11]). The **canonical complex** of M is the complex

$$\cdots \longrightarrow \Lambda^{k+1}(M) \xrightarrow{\delta} \Lambda^k(M) \xrightarrow{\delta} \Lambda^{k-1}(M) \longrightarrow \cdots$$

The homology of this complex is denoted by $H_*^{can}(M)$ and is called the **canonical homology** of M.

Remark 2.2: The canonical homology was called Poisson homology by Huebschmann [9, 10]. In [21] the Koszul operator is called the Koszul–Brylinski boundary and the canonical homology is called the (Koszul–Brylinski)–Poisson homology of M.

Remark 2.3: A remarkable feature of canonical homology is its functorial character. This means that a Poisson morphism preserves the Koszul differential and hence it induces homomorphisms in homology. Notice that the Poisson cohomology does not enjoy this property. In fact, we need to assume that the Poisson morphism is a local diffeomorphism in order to induce homomorphisms in Poisson cohomology (see [21]).

From (2) or (3) it follows that $d\delta + \delta d = 0$ [11]. Brylinski in [3] introduced the **(canonical) double complex** $\mathcal{E}_{*,*}(M)$ defined by $\mathcal{E}_{p,q}(M) = \Lambda^{q-p}(M)$, for $p, q \geq 0$, which has d for horizontal differential and δ for vertical differential (both of degree -1). This double complex is concentrated on the first quadrant. Also in [3] the **periodic double complex** $\mathcal{E}^{per}_{*,*}(M)$ is defined as follows:

$$\mathcal{E}^{per}_{p,q}(M) = \Lambda^{q-p}(M),$$

for all $p, q \in \mathbb{Z}$. Therefore (see [2, 18, 20]) there are two spectral sequences $\{E^r(M)\}$ and $\{'E^r(M)\}$ (of homological type) associated with the periodic double complex. Both of these spectral sequences converge to the total homology $H^D_*(M)$, that is, the homology of the total complex $(\mathcal{E}_k(M), D)$, where $\mathcal{E}_k(M) = \bigoplus_{p+q=k} \mathcal{E}_{p,q}(M)$ and $D = d + \delta$.

Denote by δ_r the differential of bidegree $(-r, r-1)$, so that the groups $E^{r+1}_{p,q}(M)$ are isomorphic to the homology groups of the following sequence

$$\cdots \longrightarrow E^r_{p+r,q-r+1}(M) \xrightarrow{\delta_r} E^r_{p,q}(M) \xrightarrow{\delta_r} E^r_{p-r,q+r-1}(M) \longrightarrow \cdots.$$

We must note that a differential form $\alpha \in \mathcal{E}^{per}_{p,q}(M)$ defines a class in $E^r_{p,q}(M)$ if it satisfies

$$(4)\, \delta\alpha = 0, \quad d\alpha = \delta\alpha_1, \quad d\alpha_1 = \delta\alpha_2, \quad \ldots \quad d\alpha_{r-3} = \delta\alpha_{r-2}, \quad d\alpha_{r-2} = \delta\alpha_{r-1},$$

for some differential forms $\alpha_1, \ldots, \alpha_{r-1}$. Denote by $[\alpha]_r$ the homology class defined by α in $E^r_{p,q}(M)$. The differential operator δ_r is given by

$$\delta_r[\alpha]_r = [d\alpha_{r-1}]_r. \tag{5}$$

In particular, for $r = 1$ the groups $E^1_{p,q}(M)$ of the first spectral sequence are isomorphic to the homology groups of the sequence

$$\cdots \longrightarrow \mathcal{E}^{per}_{p,q+1}(M) \xrightarrow{\delta} \mathcal{E}^{per}_{p,q}(M) \xrightarrow{\delta} \mathcal{E}^{per}_{p,q-1}(M) \longrightarrow \cdots.$$

Thus, we have

$$\begin{aligned} E^1_{p,q}(M) \cong & \frac{\{\alpha \in \mathcal{E}^{per}_{p,q}(M) | \delta\alpha = 0\}}{\delta(\mathcal{E}^{per}_{p,q+1}(M))} \\ \cong & H^{can}_{q-p}(M). \end{aligned} \tag{6}$$

For $r=2$, the groups $E^2_{p,q}(M)$ are isomorphic to the homology groups of the sequence

$$\cdots \longrightarrow E^1_{p+1,q}(M) \xrightarrow{d} E^1_{p,q}(M) \xrightarrow{d} E^1_{p-1,q}(M) \longrightarrow \cdots.$$

Then, from (6) we obtain

$$E^2_{p,q}(M) \cong \frac{\{\alpha \in \mathcal{E}^{per}_{p,q}(M) | \delta\alpha = 0 \quad \text{and} \quad d\alpha = \delta\alpha_1 \quad \text{for some } \alpha_1 \in \mathcal{E}^{per}_{p-1,q+1}(M)\}}{d(H^{can}_{q-p-1}(M))}. \tag{7}$$

Similar definitions can be given for the terms $E^r_{p,q}(M)$ ($r \geq 3$), but for our study only the groups $E^1_{p,q}(M)$ and $E^2_{p,q}(M)$ have interest.

Let $'\delta_r$ be the differential of bidegree $(r-1,-r)$, so that the groups $'E^{r+1}_{p,q}(M)$ are isomorphic to the homology groups of the sequence

$$\cdots \longrightarrow {}'E^r_{p-r+1,q+r}(M) \xrightarrow{'\delta_r} {}'E^r_{p,q}(M) \xrightarrow{'\delta_r} {}'E^r_{p+r-1,q-r}(M) \longrightarrow \cdots.$$

Now, we observe that a differential form $\beta \in \mathcal{E}^{per}_{p,q}(M)$ defines a class in $'E^r_{p,q}(M)$ if it satisfies

$$d\beta = 0, \quad \delta\beta = d\beta_1, \quad \delta\beta_1 = d\beta_2, \quad \cdots \quad \delta\beta_{r-3} = d\beta_{r-2}, \quad \delta\beta_{r-2} = d\beta_{r-1}, \tag{8}$$

for some differential forms $\beta_1, \ldots, \beta_{r-1}$. In such a case, denote by $'[\beta]_r$ the homology class defined by β in $'E^r_{p,q}(M)$. The differential $'\delta_r$ on $'E^r_{p,q}(M)$ is given by

$$'\delta'_r[\beta]_r =' [\delta\beta_{r-1}]_r. \tag{9}$$

Now, for $r=1$ the groups $'E^1_{p,q}(M)$ of the second spectral sequence are isomorphic to the homology groups of the sequence

$$\cdots \longrightarrow \mathcal{E}^{per}_{p+1,q}(M) \xrightarrow{d} \mathcal{E}^{per}_{p,q}(M) \xrightarrow{d} \mathcal{E}^{per}_{p-1,q}(M) \longrightarrow \cdots.$$

Thus we obtain

$$'E^1_{p,q}(M) \cong H^{q-p}(M), \tag{10}$$

where $H^*(M)$ denotes the de Rham cohomology of M.

To study the second spectral sequence we need the following

Lemma 2.4: *Let M be a Poisson manifold with Poisson tensor G. Then*

$$ki(G)di(G)^{k-1} = i(G)^k d + (k-1)di(G)^k, \tag{11}$$

for all $k \in \mathbb{N}$.

Proof: A general property of the Schouten–Nijenhuis bracket is that $[[i(a),d],i(b)] = i([a,b])$. Therefore, we have $[[i(G),d],i(G)] = i([G,G]) = 0$, which implies

$$2i(G)di(G) = di(G)^2 + i(G)^2 d. \tag{12}$$

Thus, (11) holds for $k = 2$. Moreover, (11) is an identity for $k = 1$.

Now, we proceed by induction. Suppose that (11) is true for all $r \leq k$. In such a case, for $r = k$ we get

$$ki(G)di(G)^{k-1} = i(G)^k d + (k-1)di(G)^k. \tag{13}$$

Applying $i(G)^{k-1}$ (on the right) to both sides of (12) we obtain $2i(G)di(G)^k = di(G)^{k+1} + i(G)^2 di(G)^{k-1}$, and hence

$$2ki(G)di(G)^k = kdi(G)^{k+1} + ki(G)^2 di(G)^{k-1}. \tag{14}$$

Applying $i(G)$ (on the left) to both sides of (13) we deduce

$$ki(G)^2 di(G)^{k-1} = i(G)^{k+1} d + (k-1)i(G)di(G)^k. \tag{15}$$

Adding (14) to (15) we have

$$(k+1)i(G)di(G)^k = i(G)^{k+1} d + kdi(G)^{k+1}. \quad \blacksquare$$

THEOREM 2.5: *For a compact Poisson manifold M, the second spectral sequence of the double complex $\mathcal{E}^{per}_{p,q}(M)$ degenerates at $'E^1(M)$, that is, $'E^1(M) \cong' E^\infty(M)$.*

Proof: We shall show that $'\delta_r = 0$ for all $r \geq 1$. Consider $\beta \in \mathcal{E}^{per}_{p,q}(M)$ such that $'[\beta]_r \in E^r_{p,q}(M)$. Hence, there exist differential forms $\beta_1, \ldots, \beta_{r-1}$ that satisfy (8). In fact, we have $\delta\beta = [i(G),d](\beta) = d(-i(G)\beta)$, because $d\beta = 0$. Thus, we can take the differential form β_1 defined by $\beta_1 = -i(G)\beta$.

Now, a direct computation using Lemma 2.4 shows that $\delta\beta_1 = d(\frac{1}{2}i(G)^2\beta)$, and hence we can define $\beta_2 = \frac{1}{2}i(G)^2\beta$. Proceeding further we obtain

$$\beta_s = \frac{(-1)^s}{s!} i(G)^s \beta, \qquad \text{for all } 1 \leq s \leq r-1.$$

Then, a representative element of the class $'\delta'_r[\beta]_r$ is

$$\delta\beta_{r-1} = \delta\left(\frac{(-1)^{r-1}}{(r-1)!} i(G)^{r-1}\beta\right) = d\left(\frac{(-1)^r}{r!} i(G)^r \beta\right),$$

which implies that $\delta\beta_{r-1}$ defines the zero homology class in $'E^r_{p+r-1,q-r}(M)$. This completes the proof. $\blacksquare$

COROLLARY 2.6: *For any compact Poisson manifold M, the total homology groups $H_k^D(M)$ are topological invariants, and they have finite dimension.*

Proof: Follows from (10) and Theorem 2.5. ∎

Let us now suppose that M is a symplectic manifold of dimension $2n$, with symplectic form ω. For any $f \in \mathfrak{F}(M)$ denote by X_f the Hamiltonian vector field corresponding to f, that is, the vector field on M such that $Z(f) = \omega(X_f, Z)$, for any $Z \in \mathfrak{X}(M)$. Then, M is a Poisson manifold whose Poisson bracket $\{,\}$ is given by $\{f, g\} = -\omega(X_f, X_g)$, for $f, g \in \mathfrak{F}(M)$. Denote by G the Poisson tensor of M, and by $\Lambda^k(G)$, $k \geq 0$, the associated pairing $\Lambda^k(G)$: $\Lambda^k(M) \times \Lambda^k(M) \longrightarrow \mathfrak{F}(M)$ which is $(-1)^k$-symmetric. Let v_M be the volume form on M given by $v_M = \omega^n/n!$. Then, imitating the Hodge star operator for Riemannian manifolds, it is possible to define the **symplectic star operator** $\star$: $\Lambda^k(M) \longrightarrow \Lambda^{2n-k}(M)$ by

$$\beta \wedge (\star\alpha) = \Lambda^k(G)(\beta, \alpha) v_M, \tag{16}$$

for $\alpha, \beta \in \Lambda^k(M)$. This operator satisfies

$$\star(\star\alpha) = \alpha, \tag{17}$$

$$\delta(\alpha) = (-1)^{k+1} \star d \star \alpha, \tag{18}$$

for all $\alpha \in \Lambda^k(M)$ (see [3, 12]). This result allows one to consider a theory of harmonic forms for symplectic (and, in general, Poisson) manifolds (see [8, 17]).

Property (18) permits us to relate the canonical homology with the de Rham cohomology of M.

THEOREM 2.7 ([3]): *Let M be a compact symplectic manifold of dimension $2n$. Then, the symplectic star operator $\star$ establishes an isomorphism of the canonical homology group $H_k^{can}(M)$ with the de Rham cohomology group $H^{2n-k}(M)$.*

Thus, the canonical homology groups of a compact symplectic manifold have finite dimension. The result is not true for arbitrary Poisson manifolds (see the next section).

Also Brylinski [3] observed the following

THEOREM 2.8: *For a compact symplectic manifold M, the first spectral sequence of the double complex $\mathcal{E}_{p,q}^{per}(M)$ degenerates at $E^1(M)$.*

Moreover, we have

THEOREM 2.9: *Let M be a compact symplectic manifold of dimension $2n$. Then, for all $r \geq 0$, the homomorphism f_r: $E^r_{p,q}(M) \longrightarrow {}'E^r_{q,2n+p}(M)$ given by $f_r[\alpha]_r =' [\star\alpha]_r$ is an isomorphism of cohomology groups. Moreover, f_r commutes with the differential, that is,*

$$(f_r \circ \delta_r)[\alpha]_r = (-1)^{q-p+1}('\delta_r \circ f_r)[\alpha]_r, \tag{19}$$

for all $[\alpha]_r \in E^r_{p,q}(M)$.

Proof: Take $\alpha \in \mathcal{E}^{per}_{p,q}(M)$ such that α defines an element of $E^r_{p,q}(M)$. Let $\alpha_1, \ldots, \alpha_{r-1}$ be differential forms on M such that they satisfy the conditions (4), that is, $\delta\alpha = 0$, $d\alpha = \delta\alpha_1, \ldots, d\alpha_{r-2} = \delta\alpha_{r-1}$.

Let us now consider $\beta = \star\alpha$. Then $\beta \in \mathcal{E}^{per}_{q,2n+p}(M)$. We define the differential forms $\beta_i = \star\alpha_i$ for all $1 \leq i \leq r-1$. From (18) it follows that $d\beta = 0$, $\delta\beta = d\beta_1, \ldots$, and $\delta\beta_{r-2} = d\beta_{r-1}$. This implies that β satisfies (8), and so β lives to ${}'E^r_{q,2n+p}(M)$. Moreover, by using (17) and again (18), we conclude that f_r is an isomorphism.

Furthermore, it is easy to check that

$$\alpha_{r-1} \in \Lambda^{q-p+2(r-1)}(M) = \mathcal{E}^{per}_{p-r+1,q+r-1}(M).$$

Thus, $\star\alpha_{r-1} \in \Lambda^{2n-q+p-2(r-1)}(M)$. Then, using (5), (9), (17), (18) and the definition of f_r, we have

$$(f_r \circ \delta_r)([\alpha]_r) = f_r[d\alpha_{r-1}]_r = {}'[\star d\alpha_{r-1}]_r, \tag{20}$$

$$('\delta_r \circ f_r)([\alpha]_r) = {}'\delta_r('[\star\alpha]_r) = {}'[\delta \star \alpha_{r-1}]_r \tag{21}$$

and

$$\delta \star \alpha_{r-1} = (-1)^{q-p+1} \star d\alpha_{r-1}. \tag{22}$$

Now, (19) follows from (20), (21) and (22). ∎

Remark 2.10: We must note that Theorem 2.8 due to Brylinski follows now from Theorem 2.5 and Theorem 2.9.

3. The canonical homology of a cosymplectic manifold

Let M be a C^∞ manifold of dimension $(2n+1)$. We say that M is a **cosymplectic manifold** (in the sense of P. Libermann [13]) if there exist a closed 2-form Φ

(called the fundamental 2-form) and a closed 1-form η on M satisfying $\Phi^n \wedge \eta \neq 0$. Then, a volume form v_M is given on M by

$$v_M = \frac{1}{n!}(\Phi^n \wedge \eta). \tag{23}$$

Also, on a cosymplectic manifold, there exists a unique global vector field $\mathcal{R}$ which satisfies $i(\mathcal{R})\Phi = 0, \quad i(\mathcal{R})\eta = 1$. The vector field $\mathcal{R}$ is called the **Reeb vector field** of the **cosymplectic structure** (Φ, η). (See [1] for an standard reference on almost contact manifolds; see also [4, 5].)

Definition 3.1: Let M be a cosymplectic manifold. For each $f \in \mathfrak{F}(M)$ the **Hamiltonian** X_f of f is the vector field on M defined by

$$\left\{ \begin{array}{rcl} \Phi(X_f, Y) & = & Y(f) - \mathcal{R}(f)\eta(Y), \\ \eta(X_f) & = & 0, \end{array} \right. \tag{24}$$

for $Y \in \mathfrak{X}(M)$.

We must remark that the existence of the vector field X_f is a consequence of the vector bundle isomorphism $\flat\colon T(M) \longrightarrow T^*(M)$ (here $T(M)$ is the tangent bundle of M, and $T^*(M)$ is the cotangent bundle of M) given by $\flat(X) = i(X)\Phi + (i(X)\eta)\eta$, where $i(X)$ denotes the interior product by X.

Because Φ is a closed 2-form on M of constant rank $2n$, $i(\mathcal{R})\Phi = 0$ and η is a closed 1-form of rank 1, from Darboux theorem it follows that there are local coordinates $\{q^1, \ldots, q^n, p_1, \ldots, p_n, z\}$ in a neighborhood of every point, such that

$$\left\{ \begin{array}{rcl} \Phi & = & \displaystyle\sum_{i=1}^n dp_i \wedge dq^i, \quad \eta = dz, \quad \mathcal{R} = \frac{\partial}{\partial z}, \\ X_f & = & \displaystyle\sum_{i=1}^n \left\{ \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} \right\}. \end{array} \right. \tag{25}$$

LEMMA 3.2: *Let M be a cosymplectic manifold, with fundamental 2-form Φ. Consider the mapping $\{,\}\colon \mathfrak{F}(M) \times \mathfrak{F}(M) \longrightarrow \mathfrak{F}(M)$ given by*

$$\{f, g\} = -\Phi(X_f, X_g), \tag{26}$$

for $f, g \in \mathfrak{F}(M)$, where X_f and X_g are the Hamiltonian vector fields of f and g, respectively. Then, $\{,\}$ is a Poisson bracket on M. Therefore, M is a Poisson manifold.

Proof: From (24) and (26) it follows that

$$\{f, g\} = -\Phi(X_f, X_g) = \Phi(X_g, X_f) = X_f(g). \tag{27}$$

Then, it is easy to check that the bracket $\{,\}$, defined by (26), satisfies the properties (i) and (ii) of a Poisson bracket. Also from (25) and (27) we obtain that $\{,\}$ satisfies the property (iii) of a Poisson bracket. ∎

Let G be the Poisson tensor on M. Then, in terms of Darboux (local) coordinates $\{q^1, \ldots, q^n, p_1, \ldots, p_n, z\}$, we have the expressions given by (25). Moreover,

$$G = \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \quad \text{and} \quad \{f, g\} = \sum_{i=1}^{n} \left\{ \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right\}. \tag{28}$$

Denote by δ the Koszul differential of M.

LEMMA 3.3: *We have:*
(i) $\delta\eta = 0$,
(ii) $i(G)(\alpha \wedge \eta) = (i(G)\alpha) \wedge \eta$,
(iii) $\delta(\alpha \wedge \eta) = (\delta\alpha) \wedge \eta$,
for any $\alpha \in \Lambda^k(M)$.

Proof: Because $\delta = [i(G), d]$ and η is closed, we obtain $\delta(\eta) = [i(G), d](\eta) = i(G)(d\eta) - d(i(G)\eta) = 0$, which proves (i). To prove (ii) we consider Darboux coordinates $\{q^1, \ldots, q^n, p_1, \ldots, p_n, z\}$ such that the 1-form η and the Poisson tensor G are given by (25) and (28), respectively. Then

$$\begin{aligned} i(G)(\alpha \wedge \eta) &= \sum_{i=1}^{n} (i(\frac{\partial}{\partial p_i}) i(\frac{\partial}{\partial q^i})(\alpha \wedge dz)) \\ &= (i(G)(\alpha)) \wedge \eta. \end{aligned}$$

This proves (ii). Now, using (ii) and $d\eta = 0$, we obtain (iii). In fact,

$$\begin{aligned} \delta(\alpha \wedge \eta) &= [i(G), d](\alpha \wedge \eta) \\ &= i(G)(d\alpha \wedge \eta) - d(i(G)(\alpha) \wedge \eta) \\ &= (i(G)(d\alpha)) \wedge \eta - d(i(G)(\alpha)) \wedge \eta \\ &= ([i(G), d](\alpha)) \wedge \eta \\ &= \delta(\alpha) \wedge \eta. \end{aligned}$$ ∎

In order to study the canonical homology of M we must consider the subspaces of $\Lambda^*(M)$ defined in the following

Definition 3.4: The subspaces $\Lambda^k_{\mathcal{R}}(M)$ and $\Lambda^k_\eta(M)$ are defined by

$$\left\{ \begin{array}{rcl} \Lambda^k_{\mathcal{R}}(M) & = & \{\alpha \in \Lambda^k(M) | i(\mathcal{R})\alpha = 0\}, \\ \Lambda^k_\eta(M) & = & \{\alpha \in \Lambda^k(M) | \eta \wedge \alpha = 0\}. \end{array} \right. \tag{29}$$

LEMMA 3.5: *For any $k \geq 1$, the space $\Lambda^k(M)$ becomes*

$$\Lambda^k(M) = \Lambda^k_{\mathcal{R}}(M) \oplus \Lambda^k_\eta(M). \tag{30}$$

Moreover, if $\alpha \in \Lambda^k(M)$ then

$$\alpha = (\alpha - \eta \wedge i(\mathcal{R})\alpha) + \eta \wedge i(\mathcal{R})\alpha, \tag{31}$$

with $(\alpha - \eta \wedge i(\mathcal{R})\alpha) \in \Lambda^k_{\mathcal{R}}(M)$ and $(\eta \wedge i(\mathcal{R})\alpha) \in \Lambda^k_\eta(M)$.

Proof: Follows directly from (29). ∎

LEMMA 3.6: *If $\alpha \in \Lambda^k_\eta(M)$ and $\beta \in \Lambda^{k-1}_{\mathcal{R}}(M)$ are such that $\alpha = \eta \wedge \beta$, then $\beta = i(\mathcal{R})\alpha$.*

Proof: From (31) it follows that $\alpha = \eta \wedge i(\mathcal{R})\alpha$. Now, let us suppose that $\beta \in \Lambda^{k-1}_{\mathcal{R}}(M)$ is such that $\alpha = \eta \wedge \beta$. This implies that $i(\mathcal{R})\alpha = i(\mathcal{R})(\eta \wedge \beta) = (i(\mathcal{R})\eta) \wedge \beta - \eta \wedge i(\mathcal{R})\beta = \beta$, because $i(\mathcal{R})\beta = 0$. ∎

The following proposition shows that δ preserves the decomposition (30).

PROPOSITION 3.7: *We have*
(i) $\delta\alpha \in \Lambda^{k-1}_{\mathcal{R}}(M)$ *for any* $\alpha \in \Lambda^k_{\mathcal{R}}(M)$;
(ii) $\delta\alpha \in \Lambda^{k-1}_\eta(M)$ *for any* $\alpha \in \Lambda^k_\eta(M)$.

Proof: Consider $\alpha \in \Lambda^k_\eta(M)$. Then, by using Lemma 3.3 (iii), we get $\eta \wedge \delta\alpha = (-1)^{k-1}\delta\alpha \wedge \eta = (-1)^{k-1}\delta(\alpha \wedge \eta) = 0$, which proves (ii). To prove (i), consider Darboux coordinates $\{q^1, \ldots, q^n, p_1, \ldots, p_n, z\}$ such that the Poisson bracket $\{,\}$ is given by (28). Now, let $\alpha \in \Lambda^k_{\mathcal{R}}(M)$. Then, α can be written

$$\alpha = \sum_{\substack{1 \leq j_1 < \cdots < j_r \leq n \\ 1 \leq i_1 < \cdots < i_s \leq n}} f^{i_1, \ldots, i_s}_{j_1, \ldots, j_r} dq^{j_1} \wedge \cdots \wedge dq^{j_r} \wedge dp_{i_1} \wedge \cdots \wedge dp_{i_s},$$

with $r + s = k$. From (3) and (28) we conclude that $\delta\alpha$ has no the 1-form dz in its local expression. Thus $i(\mathcal{R})(\delta\alpha) = 0$. ∎

Proposition 3.7 allows us to introduce the differential complexes

$$\cdots \longrightarrow \Lambda^{k+1}_{\mathcal{R}}(M) \xrightarrow{\delta} \Lambda^k_{\mathcal{R}}(M) \xrightarrow{\delta} \Lambda^{k-1}_{\mathcal{R}}(M) \longrightarrow \cdots \tag{32}$$

and

$$\cdots \longrightarrow \Lambda^{k+1}_\eta(M) \xrightarrow{\delta} \Lambda^k_\eta(M) \xrightarrow{\delta} \Lambda^{k-1}_\eta(M) \longrightarrow \cdots \tag{33}$$

each one of which is a subcomplex of the canonical complex of M. We denote by $\hat{H}{}^{can}_{*}(M)$ the homology of the complex (32), and by $\overset{\vee}{H}{}^{can}_{*}(M)$ the homology of the complex (33), that is,

$$\hat{H}{}^{can}_{k}(M) = \frac{\operatorname{Ker}\{\delta\colon \Lambda^k_{\mathcal{R}}(M) \longrightarrow \Lambda^{k-1}_{\mathcal{R}}(M)\}}{\delta(\Lambda^{k+1}_{\mathcal{R}}(M))},$$

and

$$\overset{\vee}{H}{}^{can}_{k}(M) = \frac{\operatorname{Ker}\{\delta\colon \Lambda^k_{\eta}(M) \longrightarrow \Lambda^{k-1}_{\eta}(M)\}}{\delta(\Lambda^{k+1}_{\eta}(M))}.$$

We also need to consider the restriction to $\Lambda^k_\eta(M)$ of the interior product $i(\mathcal{R})$ defined on $\Lambda^k(M)$. First, we must note that $i(\mathcal{R})(\Lambda^k_\eta(M)) \subset \Lambda^{k-1}_{\mathcal{R}}(M)$.

For any $k \geq 1$, the homomorphism $i(\overset{\wedge}{\mathcal{R}})\colon \Lambda^k_\eta(M) \longrightarrow \Lambda^{k-1}_{\mathcal{R}}(M)$ is defined by

$$i(\overset{\wedge}{\mathcal{R}})\alpha = i(\mathcal{R})\alpha, \tag{34}$$

for $\alpha \in \Lambda^k_\eta(M)$.

Proposition 3.8: *For any $k \geq 1$, the homomorphism $i(\overset{\wedge}{\mathcal{R}})$ given by (34) satisfies*

$$\delta(i(\overset{\wedge}{\mathcal{R}})\alpha) = -i(\overset{\wedge}{\mathcal{R}})(\delta\alpha), \tag{35}$$

for $\alpha \in \Lambda^k_\eta(M)$.

Proof: Because $\alpha \in \Lambda^k_\eta(M)$, from (31) it follows that $\alpha = \eta \wedge i(\mathcal{R})\alpha$. Then, by using Lemma 3.3 (iii) and (34), we have $\delta\alpha = \delta(\eta \wedge (i(\mathcal{R})\alpha)) = -\eta \wedge \delta(i(\mathcal{R})\alpha) = -\eta \wedge \delta(i(\overset{\wedge}{\mathcal{R}})\alpha)$. Now, (35) follows from Lemma 3.6. ∎

As a consequence of (35) we obtain that the homomorphism $i(\overset{\wedge}{\mathcal{R}})$ given by (34) induces the homomorphism $i(\overset{\wedge}{\mathcal{R}})\colon \overset{\vee}{H}{}^{can}_{k}(M) \longrightarrow \hat{H}{}^{can}_{k-1}(M)$ defined by

$$i(\overset{\wedge}{\mathcal{R}})\{\alpha\} = \{i(\mathcal{R})\alpha\}, \tag{36}$$

where $\{\alpha\}$ denotes the homology class in $\overset{\vee}{H}{}^{can}_{k}(M)$ defined by the k-form $\alpha \in \Lambda^k_\eta(M)$ that satisfies $\delta\alpha = 0$, and $\{i(\mathcal{R})\alpha\}$ denotes the homology class in $\hat{H}{}^{can}_{k-1}(M)$ defined by $i(\mathcal{R})\alpha$.

Proposition 3.9: *The homomorphism $i(\overset{\wedge}{\mathcal{R}})\colon \overset{\vee}{H}{}^{can}_{k}(M) \longrightarrow \hat{H}{}^{can}_{k-1}(M)$ given by (36) is an isomorphism.*

Proof: Consider $\alpha \in \Lambda^k_\eta(M)$ such that $\delta\alpha = 0$. Suppose that $i(\mathcal{R})\alpha$ defines the zero homology class in $\hat{H}{}^{can}_{k-1}(M)$. This implies that $i(\mathcal{R})\alpha = \delta\beta$ for some $\beta \in \Lambda^k_{\mathcal{R}}(M)$. Then $\eta\wedge\beta \in \Lambda^{k+1}_\eta(M)$ and we have $\alpha = \eta\wedge i(\mathcal{R})\alpha = \eta\wedge\delta\beta = \delta(-\eta\wedge\beta)$, that is, α defines also the zero homology class, and so $\overset{\wedge}{i(\mathcal{R})}$ is injective.

Consider $\beta \in \Lambda^{k-1}_{\mathcal{R}}(M)$ a representative element of a class $a \in \hat{H}{}^{can}_{k-1}(M)$. Define $\alpha = \eta \wedge \beta$. Then $\alpha \in \Lambda^k_\eta(M)$, and moreover $\delta\alpha = 0$. Now, by using Lemma 3.6 we conclude that $\beta = i(\mathcal{R})\alpha$. Thus $\overset{\wedge}{i(\mathcal{R})}\{\alpha\} = \{\beta\} = a$, which proves that $\overset{\wedge}{i(\mathcal{R})}$ is surjective. ∎

From (30), (31), Proposition 3.7 and Proposition 3.9, we obtain the isomorphisms

$$H^{can}_k(M) \cong \hat{H}{}^{can}_k(M) \oplus \overset{\vee}{H}{}^{can}_k(M) \tag{37}$$

$$\cong \hat{H}{}^{can}_k(M) \oplus \hat{H}{}^{can}_{k-1}(M), \tag{38}$$

for any $k \geq 1$.

In order to study $\hat{H}{}^{can}_k(M)$ we need introduce the following

Definition 3.10: For $k \geq 0$, the map $d_{\mathcal{R}}\colon \Lambda^k_{\mathcal{R}}(M) \longrightarrow \Lambda^{k+1}_{\mathcal{R}}(M)$ is defined by

$$d_{\mathcal{R}}\alpha = d\alpha - \eta \wedge i(\mathcal{R})d\alpha, \tag{39}$$

for $\alpha \in \Lambda^k_{\mathcal{R}}(M)$.

PROPOSITION 3.11: *We have*

$$d_{\mathcal{R}}(\alpha \wedge \beta) = (d_{\mathcal{R}}\alpha) \wedge \beta + (-1)^k \alpha \wedge d_{\mathcal{R}}\beta, \tag{40}$$

for $\alpha \in \Lambda^k_{\mathcal{R}}(M)$ and $\beta \in \Lambda^*_{\mathcal{R}}(M)$; and

$$d^2_{\mathcal{R}} = 0. \tag{41}$$

Proof: Because $i(\mathcal{R})\alpha = i(\mathcal{R})\beta = 0$, from (39) we get

$$\begin{aligned} d_{\mathcal{R}}(\alpha \wedge \beta) &= d(\alpha \wedge \beta) - \eta \wedge i(\mathcal{R})(d(\alpha \wedge \beta)) \\ &= d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta - \eta \\ &\qquad\qquad \wedge \left\{(i(\mathcal{R})d\alpha) \wedge \beta + (-1)^k(-1)^k \alpha \wedge (i(\mathcal{R})d\beta)\right\} \\ &= (d_{\mathcal{R}}\alpha) \wedge \beta + (-1)^k \alpha \wedge (d_{\mathcal{R}}\beta), \end{aligned}$$

which proves (40). Similarly, by using that $d\eta = 0$, from (39) we obtain (41). ∎

Thus, we have the differential complex

$$\cdots \longrightarrow \Lambda^{k-1}_{\mathcal{R}}(M) \xrightarrow{d_{\mathcal{R}}} \Lambda^{k}_{\mathcal{R}}(M) \xrightarrow{d_{\mathcal{R}}} \Lambda^{k+1}_{\mathcal{R}}(M) \longrightarrow \cdots. \tag{42}$$

Denote by $H^*_{\mathcal{R}}(M)$ the cohomology of (42).

Remark 3.12: In general, the differential complex $(\Lambda^*_{\mathcal{R}}(M), d_{\mathcal{R}})$ is not elliptic. Indeed, the symbol of the differential operator $d_{\mathcal{R}}$ is $\sigma_1(d_{\mathcal{R}})(x,\mu)(\alpha) = (\mu - \mu(\mathcal{R}(x))\eta(x)) \wedge \alpha$, for all 1-form non-zero μ of $\Lambda^1_{\mathcal{R}}(M)$ and for all k-form α at x. Thus, the complex

$$\cdots \longrightarrow (\Lambda^{k-1}_{\mathcal{R}})_x(M) \xrightarrow{(\mu-\mu(\mathcal{R}(x)))\wedge} (\Lambda^{k}_{\mathcal{R}})_x(M) \xrightarrow{(\mu-\mu(\mathcal{R}(x)))\wedge} (\Lambda^{k+1}_{\mathcal{R}})_x(M) \longrightarrow \cdots$$

is not exact if $\mu = \eta(x)$. As a consequence, we are not able to prove if the canonical homology groups are finite. However, if we take $M = N \times S^1$, where N is a compact symplectic manifold with symplectic form ω and define a cosymplectic structure (Φ, η) on M, where $\Phi = \omega$ and η is the standard volume form on S^1 (with the obvious identifications), it is not hard to prove that the canonical homology groups of M have infinite dimension.

Next, we shall prove a "similar" result to Theorem 2.7 for the canonical homology $H^{can}_*(M)$ of any compact cosymplectic manifold. We proceed as follows. First, imitating the definition given by (16) of the symplectic star operator for symplectic manifolds, we define the **cosymplectic $\mathcal{R}$-star operator** $\star_{\mathcal{R}}\colon \Lambda^k_{\mathcal{R}}(M) \longrightarrow \Lambda^{2n-k}_{\mathcal{R}}(M)$ by the condition

$$\beta \wedge (\star_{\mathcal{R}}\alpha) = \Lambda^k G(\beta,\alpha) v^{\mathcal{R}}_M,$$

for $\alpha, \beta \in \Lambda^k_{\mathcal{R}}(M)$, and where $v^{\mathcal{R}}_M = i(\mathcal{R})v_M$, and v_M is defined by (23).

Moreover, we consider the operator $\delta_{\mathcal{R}}\colon \Lambda^k_{\mathcal{R}}(M) \longrightarrow \Lambda^{k-1}_{\mathcal{R}}(M)$ given by

$$\delta_{\mathcal{R}} = [i(G), d_{\mathcal{R}}]. \tag{43}$$

Then, the same proofs given by Brylinski for (17) and (18) show that

$$\star_{\mathcal{R}}(\star_{\mathcal{R}}\alpha) = \alpha, \tag{44}$$

$$\delta_{\mathcal{R}}(\alpha) = (-1)^{k+1} \star_{\mathcal{R}} d_{\mathcal{R}} \star_{\mathcal{R}} (\alpha), \tag{45}$$

for $\alpha \in \Lambda^k_{\mathcal{R}}(M)$.

Furthermore, we have

PROPOSITION 3.13: *Let M be a cosymplectic manifold. Then the Koszul differential δ of M satisfies*

$$\delta\alpha = [i(G), d_{\mathcal{R}}](\alpha), \tag{46}$$

for $\alpha \in \Lambda^k_{\mathcal{R}}(M)$ and $k \geq 0$.

Proof: Consider $\alpha \in \Lambda^k_{\mathcal{R}}(M)$. From Proposition 3.7(i) it follows that $\delta\alpha \in \Lambda^{k-1}_{\mathcal{R}}(M)$, and hence $i(\mathcal{R})\delta\alpha = 0$. Then, by using (39) and Lemma 3.3 (ii), we obtain

$$\begin{aligned}[i(G), d_{\mathcal{R}}](\alpha) &= i(G)(d\alpha - \eta \wedge i(\mathcal{R})d\alpha) - (d(i(G)\alpha) - \eta \wedge i(\mathcal{R})d(i(G)\alpha)) \\ &= [i(G), d](\alpha) - \eta \wedge (i(\mathcal{R})\delta\alpha) \\ &= [i(G), d](\alpha) = \delta\alpha. \quad ∎\end{aligned}$$

Now, from (43), (45) and (46) it follows that

$$\delta\alpha = (-1)^{k+1} \star_{\mathcal{R}} d_{\mathcal{R}} \star_{\mathcal{R}} (\alpha), \tag{47}$$

for $\alpha \in \Lambda^k_{\mathcal{R}}(M)$.

COROLLARY 3.14: *The cosymplectic $\mathcal{R}$-star operator $\star_{\mathcal{R}}$ establishes an isomorphism of the cohomology group $H^k_{\mathcal{R}}(M)$ with the homology group $\hat{H}{}^{can}_{2n-k}(M)$ of any cosymplectic manifold M of dimension $(2n+1)$. Therefore, there is also an isomorphism*

$$H^{can}_k(M) \cong H^{2n-k}_{\mathcal{R}}(M) \oplus H^{2n-k+1}_{\mathcal{R}}(M) \tag{48}$$

Proof: Follows directly from (44) and (47). ∎

4. The canonical homology of compact cosymplectic nilmanifolds

In this section we show an "approximation to Nomizu theorem" for the canonical homology of compact cosymplectic nilmanifolds. Indeed, Nomizu theorem is a useful tool to compute the de Rham cohomology of a compact nilmanifold, and it was successfully used in [6] to exhibit a large family of examples of symplectic nilmanifolds without Kähler structure.

Let us consider a compact nilmanifold M of dimension $(2n+1)$, that is, M is a quotient space $M = \Gamma\backslash K$, where K is a connected, simply-connected and nilpotent Lie group of dimension $(2n+1)$, and Γ is a discrete subgroup of K such that the quotient space $M = \Gamma\backslash K$ is compact. (Let us recall that such a subgroup

Γ exists if and only if the nilpotent Lie group K is rational, that is, there exists a basis of left invariant 1-forms such that the coefficients in the structure equations are rational numbers [16].) It is easy to see that all left invariant tensor fields on K descend to M. For convenience, if μ is a left invariant tensor field on K, we shall also denote by μ the tensor field induced on M.

Next, we consider that $M = \Gamma\backslash K$ is a compact manifold with a cosymplectic structure (Φ, η) that arises from a left invariant cosymplectic structure on K. Then, from Lemma 3.2, we know that K and M are Poisson manifolds. Denote by δ the Koszul differential of K and M.

Let $\mathfrak{K}$ be the Lie algebra of K; and let

$$\cdots \longrightarrow \Lambda^{q-1}(\mathfrak{K}^*) \xrightarrow{d} \Lambda^{q}(\mathfrak{K}^*) \xrightarrow{d} \Lambda^{q+1}(\mathfrak{K}^*) \longrightarrow \cdots$$

be the Chevalley–Eilenberg complex, where $\Lambda^q(\mathfrak{K}^*)$ is the space of left invariant differential q-forms on K. Denote by $H^*(\mathfrak{K}^*)$ the Chevalley–Eilenberg cohomology.

Moreover, we can consider the differential complex

$$\cdots \longrightarrow \Lambda^{q+1}(\mathfrak{K}^*) \xrightarrow{\delta} \Lambda^{q}(\mathfrak{K}^*) \xrightarrow{\delta} \Lambda^{q-1}(\mathfrak{K}^*) \longrightarrow \cdots. \tag{49}$$

Denote by $H_*^{can}(\mathfrak{K}^*)$ the homology of the complex (49).

Now, for $q \geq 0$, we need consider the subspace $\Lambda^q_{\mathcal{R}}(\mathfrak{K}^*)$ of $\Lambda^q(\mathfrak{K}^*)$, and the map $d_{\mathcal{R}}\colon \Lambda^q_{\mathcal{R}}(\mathfrak{K}^*) \longrightarrow \mathfrak{f}^{q+1}_{\mathcal{R}}(\mathfrak{K}^*)$, which we define by similar relations to (29) and (39), respectively. Thus, we have the differential complex

$$\cdots \longrightarrow \Lambda^{q-1}_{\mathcal{R}}(\mathfrak{K}^*) \xrightarrow{d_{\mathcal{R}}} \Lambda^{q}_{\mathcal{R}}(\mathfrak{K}^*) \xrightarrow{d_{\mathcal{R}}} \Lambda^{q+1}_{\mathcal{R}}(\mathfrak{K}^*) \longrightarrow \cdots. \tag{50}$$

The complex (50) is a differential subcomplex of the complex $(\Lambda^*_{\mathcal{R}}(M), d_{\mathcal{R}})$. Denote by $H^*_{\mathcal{R}}(\mathfrak{K}^*)$ the cohomology of the complex (50).

From (48) we get that there is an isomorphism of groups

$$H_q^{can}(\mathfrak{K}^*) \cong H^{2n-q}_{\mathcal{R}}(\mathfrak{K}^*) \oplus H^{2n-q+1}_{\mathcal{R}}(\mathfrak{K}^*). \tag{51}$$

In order to prove an approximation to Nomizu theorem for the canonical homology $H_*^{can}(M)$ of the compact nilmanifold M, we shall show that there exists an injective homomorphism of $H^q_{\mathcal{R}}(\mathfrak{K}^*)$ into $H^q_{\mathcal{R}}(M)$. For this, we need to consider the Lie subalgebra $\mathfrak{H}$ of $\mathfrak{K}$ defined by

$$\mathfrak{H} = \{X \in \mathfrak{K} | \eta(X) = 0\}.$$

Because $\mathfrak{K}$ is a rational nilpotent Lie algebra of dimension $(2n+1)$, we see that $\mathfrak{H}$ is a rational nilpotent Lie algebra of dimension $2n$. Therefore, we know that there exists a connected, simply-connected and nilpotent Lie group H whose Lie algebra is $\mathfrak{H}$.

Then, Mal'cev theorem [16] implies that there exists a discrete subgroup $\widetilde{\Gamma}$ of H such that the quotient space $N = \widetilde{\Gamma}\backslash H$ is compact. Moreover [7], such a subgroup $\widetilde{\Gamma}$ is $\widetilde{\Gamma} = \Gamma \cap H$.

Denote by $\widetilde{d}$ the exterior differential of N. For all $q \geq 0$, it is easy to see that

$$\Lambda^q_{\mathcal{R}}(\mathfrak{K}^*) = \Lambda^q(\mathfrak{H}^*), \tag{52}$$

$$d_{\mathcal{R}}(\alpha) = \widetilde{d}(\alpha), \tag{53}$$

for $\alpha \in \Lambda^q(\mathfrak{H}^*)$. Now, (52), (53) and Nomizu theorem imply that there exist the isomorphisms

$$H^q_{\mathcal{R}}(\mathfrak{K}^*) \cong H^q(\mathfrak{H}^*) \cong H^q(N), \tag{54}$$

for $q \geq 0$.

Let $j\colon N = \widetilde{\Gamma}\backslash H \longrightarrow M = \Gamma\backslash K$ be the natural inclusion. Because j is a differentiable map, we can consider the map $j^*\colon \Lambda^q(M) \longrightarrow \Lambda^q(N)$. Denote the natural inclusion by $i\colon \Lambda^q_{\mathcal{R}}(M) \longrightarrow \Lambda^q(M)$. We define the map $\lambda\colon \Lambda^q_{\mathcal{R}}(M) \longrightarrow \Lambda^q(N)$ as the composition of i and j^*,

$$\lambda(\alpha) = j^*(i(\alpha)) = j^*(\alpha), \tag{55}$$

for $\alpha \in \Lambda^q_{\mathcal{R}}(M)$. We must note that the induced map $\lambda\colon \Lambda^q_{\mathcal{R}}(\mathfrak{K}^*) \longrightarrow \Lambda^q(\mathfrak{H}^*)$ is the identity map, that is,

$$\lambda(\alpha) = \alpha \quad \text{for } \alpha \in \Lambda^q_{\mathcal{R}}(\mathfrak{K}^*). \tag{56}$$

Furthermore, for $\alpha \in \Lambda^q_{\mathcal{R}}(M)$, from (55) and (39) we obtain

$$\begin{aligned} \widetilde{d}(\lambda(\alpha)) &= \widetilde{d}(j^*\alpha) = j^*(d\alpha) \\ &= j^*(d_{\mathcal{R}}(\alpha) + \eta \wedge i(\mathcal{R})(d\alpha)) \\ &= j^*(d_{\mathcal{R}}(\alpha)) = \lambda(d_{\mathcal{R}}(\alpha)), \end{aligned}$$

which means that λ commutes with the differential. Thus, we have

$$\widetilde{d} \circ \lambda = \lambda \circ d_{\mathcal{R}}. \tag{57}$$

THEOREM 4.1: *Let $M = \Gamma\backslash K$ be a compact cosymplectic nilmanifold, whose cosymplectic structure arises from a left invariant cosymplectic structure on K. Let $\mu\colon (\Lambda^*_{\mathcal{R}}(\mathfrak{K}^*), d_{\mathcal{R}}) \longrightarrow (\Lambda^*_{\mathcal{R}}(M), d_{\mathcal{R}})$ be the natural inclusion. Then, the induced homomorphism $\overline{\mu}\colon H^q_{\mathcal{R}}(\mathfrak{K}^*) \longrightarrow H^q_{\mathcal{R}}(M)$ is injective.*

Proof: Consider $\alpha \in \Lambda^q_{\mathcal{R}}(\mathfrak{K}^*)$ such that $d_{\mathcal{R}}(\alpha) = 0$. Denote by $[\alpha]_{\mathcal{R}}$ the cohomology class defined by α in $H^q_{\mathcal{R}}(\mathfrak{K}^*)$ and $H^q_{\mathcal{R}}(M)$. Suppose that $[\alpha]_{\mathcal{R}}(=\overline{\mu}[\alpha]_{\mathcal{R}})$ is the zero cohomology class in $H^q_{\mathcal{R}}(M)$. This implies that

$$\alpha = d_{\mathcal{R}}(\beta), \tag{58}$$

for some $\beta \in \Lambda^{q-1}_{\mathcal{R}}(M)$.

Applying λ to both sides of (58), and using (56) and (57), we get

$$\alpha = \lambda(\alpha) = \lambda(d_{\mathcal{R}}\beta) = \widetilde{d}(\lambda\beta). \tag{59}$$

From (52), (54) and (59) it follows that $\alpha \in \Lambda^q(\mathfrak{H}^*)$ and it defines the zero cohomology class in $H^q(N)$. Thus, α defines the zero cohomology class in $H^q_{\mathcal{R}}(\mathfrak{K}^*)$. This completes the proof. ∎

COROLLARY 4.2: *Let $M = \Gamma\backslash K$ be a compact cosymplectic nilmanifold, whose cosymplectic structure arises from a left invariant one on K. Then, there is an injective homomorphism ν: $H^{can}_q(\mathfrak{K}^*) \longrightarrow H^{can}_q(M)$, for all $q \geq 0$.*

Proof: It follows directly from (51) and Theorem 4.1. ∎

5. The compact cosymplectic nilmanifold M^5

In this section, we construct an example of a compact cosymplectic nilmanifold M^5 whose first spectral sequence $\{E^r(M^5)\}$ does not degenerates at the first term.

Let K be the 5-dimensional connected, simply-connected and nilpotent Lie group, defined by the left invariant 1-forms $\{\alpha_1, \ldots, \alpha_5\}$ such that

$$d\alpha_1 = d\alpha_2 = d\alpha_5 = 0, \quad d\alpha_3 = \alpha_2 \wedge \alpha_5, \quad d\alpha_4 = \alpha_1 \wedge \alpha_2. \tag{60}$$

The structure equations (60) can be integrated explicitly; in fact, K can be realized as the nilpotent Lie group

$$K = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & x_5 & x_3 & x_4 \\ 0 & 1 & 0 & 0 & 0 & -x_2 \\ 0 & 0 & 1 & 0 & -x_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x_i \in \mathbb{R} \right\}.$$

Because the coefficients in the structure equations (60) are integers, Mal'cev theorem [16] implies that K has a uniform subgroup Γ. We take Γ to be the subgroup of K consisting of those matrices whose entries are integers. Define $M^5 = \Gamma \backslash K$.

We consider the cosymplectic structure on M^5 given by

$$\Phi = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3, \quad \eta = \alpha_5.$$

Therefore, the Reeb vector field is $\mathcal{R} = X_5$, for $\{X_1, \ldots, X_5\}$ the basis dual to $\{\alpha_1, \ldots, \alpha_5\}$.

Now, the vector bundle isomorphism $\flat : T(M^5) \longrightarrow T^*(M^5)$ is given by $\flat(X_1) = \alpha_4$, $\flat(X_2) = \alpha_3$, $\flat(X_3) = -\alpha_2$, $\flat(X_4) = -\alpha_1$, $\flat(X_5) = \eta$. Moreover, the Poisson tensor $G = -\flat^{-1}(\Phi)$ is

$$G = -X_1 \wedge X_4 - X_2 \wedge X_3.$$

It is easy to verify that, for the Koszul differential δ of M^5 with Poisson tensor G, $\delta(\alpha_3 \wedge \alpha_4) = \alpha_1$, $\delta(\alpha_2 \wedge \alpha_3 \wedge \alpha_4) = -\alpha_1 \wedge \alpha_2$, $\delta(\alpha_3 \wedge \alpha_4 \wedge \eta) = \alpha_1 \wedge \eta$, $\delta(\alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \eta) = -\alpha_1 \wedge \alpha_2 \wedge \eta$, and $\delta(\beta) = 0$ for the another left invariant forms β.

Denote by $\mathfrak{K}$ the Lie algebra of K. Then, we have

$$\begin{aligned}
H_0^{can}(\mathfrak{K}^*) =& \{1\}, \\
H_1^{can}(\mathfrak{K}^*) =& \{\{\alpha_2\}, \{\alpha_3\}, \{\alpha_4\}, \{\eta\}\}, \\
H_2^{can}(\mathfrak{K}^*) =& \{\{\alpha_1 \wedge \alpha_3\}, \{\alpha_1 \wedge \alpha_4\}, \{\alpha_2 \wedge \eta\}, \{\alpha_2 \wedge \alpha_3\}, \{\alpha_2 \wedge \alpha_4\}, \\
& \{\alpha_3 \wedge \eta\}, \{\alpha_4 \wedge \eta\}\}, \\
H_3^{can}(\mathfrak{K}^*) =& \{\{\alpha_1 \wedge \alpha_2 \wedge \alpha_3\}, \{\alpha_1 \wedge \alpha_2 \wedge \alpha_4\}, \{\alpha_1 \wedge \alpha_3 \wedge \eta\}, \\
& \{\alpha_1 \wedge \alpha_3 \wedge \alpha_4\}, \{\alpha_1 \wedge \alpha_4 \wedge \eta\}, \{\alpha_2 \wedge \alpha_3 \wedge \eta\}, \{\alpha_2 \wedge \alpha_4 \wedge \eta\}\}, \\
H_4^{can}(\mathfrak{K}^*) =& \{\{\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4\}, \{\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \eta\}, \{\alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \eta\}, \\
& \{\alpha_1 \wedge \alpha_3 \wedge \alpha_4 \wedge \eta\}\}, \\
H_5^{can}(\mathfrak{K}^*) =& \{\{\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \eta\}\},
\end{aligned}$$

where we denote by $\{\alpha_2\}$ the homology class defined by α_2 and so forth.

Thus, using Corollary 4.2, we get

$$\dim(H_1^{can}(M^5)) \geq 4. \tag{61}$$

THEOREM 5.1: *For the first spectral sequence $\{E^1(M^5)\}$ we have*

$$E^1_{0,1}(M^5) \not\cong E^2_{0,1}(M^5).$$

Proof: Consider the 1-form α_3. We know that α_3 defines a nontrivial homology class $\{\alpha_3\}$ in $H_1^{can}(\mathfrak{K}^*)$. Then, by Corollary 4.2, α_3 defines a nontrivial homology class in $H_1^{can}(M^5)$. Moreover, from (6) it follows that $E_{0,1}^1(M^5) \cong H_1^{can}(M^5)$, so that α_3 represents a nontrivial class in $E_{0,1}^1(M^5)$.

However, $d\alpha_3 = \alpha_2 \wedge \eta$; and we know that $\alpha_2 \wedge \eta$ defines a nontrivial class in $H_2^{can}(\mathfrak{K}^*)$. Therefore, using Corollary 4.2 again, we obtain that $\alpha_2 \wedge \eta$ represents a nontrivial homology class in $H_2^{can}(M^5)$. This implies that

$$\alpha_2 \wedge \eta \notin \delta(\Lambda^3(M^5)). \tag{62}$$

From (7) and (62) we see that α_3 does not live in $E_{0,1}^2(M^5)$, and therefore $\delta_1 \neq 0$, which finishes the proof. ∎

Finally, we show that for the compact Poisson manifold M^5 also fails Theorem 2.7. From Nomizu theorem [19] we compute the de Rham cohomology groups $H^q(M^5)$ of M^5. They are:

$$\begin{aligned}
H^0(M^5) =& \{1\},\\
H^1(M^5) =& \{[\alpha_1], [\alpha_2], [\eta]\},\\
H^2(M^5) =& \{[\alpha_1 \wedge \alpha_3 + \alpha_4 \wedge \eta], [\alpha_1 \wedge \alpha_4], [\alpha_1 \wedge \eta], [\alpha_2 \wedge \alpha_3],\\
& [\alpha_2 \wedge \alpha_4], [\alpha_3 \wedge \eta]\},\\
H^3(M^5) =& \{[\alpha_1 \wedge \alpha_2 \wedge \alpha_3], [\alpha_1 \wedge \alpha_2 \wedge \alpha_4], [\alpha_1 \wedge \alpha_3 \wedge \eta],\\
& [\alpha_1 \wedge \alpha_4 \wedge \eta], [\alpha_2 \wedge \alpha_3 \wedge \alpha_4], [\alpha_2 \wedge \alpha_3 \wedge \eta]\},\\
H^4(M^5) =& \{[\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4], [\alpha_1 \wedge \alpha_3 \wedge \alpha_4 \wedge \eta], [\alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \eta]\},\\
H^5(M^5) =& \{[\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \eta]\}.
\end{aligned}$$

Therefore, using (61), we find

$$\dim H^4(M^5) = 3 < 4 \leq \dim H_1^{can}(M^5),$$

which proves the following

THEOREM 5.2: *We have*

$$H_1^{can}(M^5) \ncong H^4(M^5).$$

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